**Risk and Risk-Aversion Part 2**

**From Bodie, Kane and Marcus Chapter 6**

**One risk-free asset and one risky asset**

Let’s form a portfolio from the risk-free and risky assets in Part 1. Recall that the risky investment has an expected return of 22%, and a standard deviation of 34.29%. The risk-free asset has a return of 5%, and of course, a standard deviation of zero.

Let’s invest 60% of our money in the risky asset and the other 40% in the risk-free asset. This means that the weight on the risky asset will be 0.6 and the weight on the risk-free asset will be 0.4. What is the expected return and standard deviation of this portfolio?

The expected return of any portfolio is a weighted average of the expected returns of the assets that make up the portfolio – with each asset being weighted by the proportion of our money that we invest in it.

E(RP) = w1 ∙ E(R1) + w2 ∙ E(R2) + w3 ∙ E(R3) + …

So for a portfolio comprised of one risky asset and one risk-free asset:

E(RP) = wrisky ∙ E(Rrisky) + wRf ∙ Rf

In this equation, wrisky and wRf are the weights (or proportions or fractions) of the portfolio invested in the risky and risk-free assets respectively. Since the fractions of the total invested in the risky and risk-free assets have to add up to 1 or 100%,

we have wRf = 1 - wrisky. So, we will drop the subscript and simply use w for the fraction of the portfolio invested in the risky asset, implying that the proportion of the total invested in the risk-free asset is (1-w).

And since we are using “P” as the subscript for the portfolio, we can eliminate the subscript for the risky asset.

With these simplifications, the above equation reduces to:

(1) Portfolio expected return = E(RP) = w∙ E(R) + (1 – w) ∙ Rf

Equation (1) can also be written as:

(1A) E(RP) = Rf + w(E(R) – Rf)

There is a neat interpretation that we can provide for equation (1A): The base rate of return for any portfolio is the risk-free rate, Rf. In addition, the portfolio is expected to return a risk premiumthat depends upon the risk premium of the risky asset, (E(R) – Rf), and the fraction of the portfolio invested in the risky asset (w)

The standard deviation of a portfolio is the square root of its variance.

The variance of a portfolio though, is **not** (normally) a weighted average of the variances of the assets that make up the portfolio.

The variance of a portfolio is the sum of the cells in a weighted variance/covariance matrix.

However, when there are only two assets in the portfolio, and one of them is the risk-free asset, this becomes rather simple:

Since the covariance between the risk-free asset and any other asset is zero, and the risk-free asset has a standard deviation of zero, the standard deviation of a portfolio which includes a risky asset and the risk-free asset reduces to:

(2) σP = w ∙ σ

To return to our example, *w*=0.60, 1-*w*=0.40, E(R)= 22%, Rf = 5%, and σ = 34.29%.

Now, we can substitute into the above equations:

E(RP) = (0.6) (22%) + (0.4) (5%) = 15.2%, using equation (1)

σP = 0.6 x 34.29% = 20.57%, using equation (2)

Of course, we can also see that 15.2% = 5% + 0.6(22% - 5%), using equation (1A), and the excess return interpretation.

**Back to our investor**

What does this mean for our investor with a coefficient of risk aversion, A=3.0?

Recall that given a choice between either the risk-free asset, or the risky asset, we concluded that this investor would be better off (have a higher utility) investing in the risk-free asset.

We can now calculate the utility she gets from our 60:40 portfolio of the risky and risk-free assets.

Since the portfolio has an expected return of 15.2%, and a standard deviation 20.57%, we can calculate that

UP= .152 – (1/2) (3) (.2057)2) = 8.85%.

Obviously, our investor with A=3.0 would prefer this portfolio over either asset alone (remember that UR = 4.36% and URf = 5%).

Clearly 60:40 is not the only feasibleportfolio combination. There are many others (in fact an infinite number of combinations are possible). Any pair of real numbers (including negative numbers!) that add up to 1 is just fine to plug in as the weights.

This suggests the following two-step decision-making procedure for our investor.

Step 1: Generate the feasible setof all possible combinations of the risk-free asset and the risky asset.

Step 2: From all such feasible combinations, choose the one combination that maximizes her utility. This particular combination is her optimal portfolio.

**The Capital Allocation Line**

The Capital Allocation Line (CAL) plots all feasible portfolio combinations of the risky and risk-free assets.

To derive the equation of the CAL, we need to algebraically relate the risk and return

i.e. standard deviation and expected return of the portfolio.

We can rewrite equation (2) as: , and substitute this value of w into equation (1A) to yield:

E(RP) = Rf + w(E(R) – Rf) = Rf + (E(R) – Rf) 

This is usually written as:

 E(RP) = Rf +  σP

This is the equation of the Capital Allocation Line.

For our example, the equation of the CAL reads:

E(RP) = .05 + σP

From high school math, we know this equation is of the form:

 y=mx+b,

which is the equation of a straight line, with m as its slope and b the vertical intercept. This is why, mathematically speaking, the CAL is a straight line in portfolio expected return/standard deviation space

**Graph of the CAL**

To gain some better intuition, let’s draw a graph of the CAL for our example. Since we know that the CAL is a straight line, we only need two points to draw it. Let’s choose the two easiest points that we know mustbe on this line:

1. A portfolio of 100% risky asset, 0% risk-free asset, in other words, the risky asset itself. E(RP) = 22% and σP = 34.29%
2. A portfolio of 0% risky asset, 100% risk-free asset, in other words, the risk-free asset itself. E(RP) = 5% with σP = 0



The infinite number of points that make up this line correspond to the infinite number of feasible combinations of the risky and the risk-free assets in a portfolio.

Among all these points, there are two points that interest us more than the others: the little black square in the above graph, which is our risky asset, and the little black circle, which is our risk-free asset. The latter plots on the vertical axis, as its standard deviation is zero.

The slope of the CAL is m = Rise/Run = .

This quantity can be interpreted as the excess expected return per unit of standard deviation. In other words, it is a reward-to-risk ratio.

In finance, this term is known as the Sharpe Ratio.

Before we determine the optimal portfolio for our investor, we should talk about the little triangle and the little diamond on the above figure, to get some more insight into the CAL.

**Lending and borrowing portfolios**

Let’s say (continuing with our risky and risk-free assets) that Liza wants a portfolio with an expected return of 15%. How would she go about forming such a portfolio?

If she knows equation (1), she can solve for the proportions as follows:

E(RP) = w∙ E(R) + (1 – w) Rf = 15%

 = w∙ 22% + (1 – w) ∙ 5% = 15%

.22w + .05 - .05w = .15

.17w = .10

w = .10/.17 = 58.82%

Liza invests 58.82% of her money in the risky asset, and the remaining 41.18% in the risk-free asset. If she has $100 to invest, she invesets $58.82 in our risky asset and $41.18 in the risk-free asset.

The standard deviation of her portfolio can be found using equation (2):

= (0.5882) (34.29%) = 20.17%.

The little black triangle on the graph represents Liza’s portfolio in mean/variance space.

Now, imagine another investor Barb who wants a portfolio with an expected return of 30%. How would she go about forming such a portfolio?

Again, she uses equation (1) to solve for the proportions as follows:

E(RP) = w∙ E(R) + (1 – w) Rf = 30%

 = w∙ 22% + (1 – w) ∙ 5% = 30%

.22w + .05 - .05w = .30

.17w = .25

w = .25/.17 = 147.06%

Therefore, 1-w = -47.06%

How do we interpret the negative weight on the risk-free asset?

Remember that a positive weight means investing in the risk-free asset, i.e. *lending money* to someone (most likely the U.S. Govt. by buying T-bills) and getting the risk-free rate of return. The converse of this statement says that a negative weight must mean *borrowing* money at the risk-free rate.

This means that if Barb has $100, she needs to do the following:

Borrow $47.06 (at the risk-free rate of 5%), add this to her own $100, and invest the entire $147.06 in the risky asset.

This enables her to obtain a higher expected return (30%) than by investing all her money in the risky asset (which would have an expected return of 22%).

This is called a leveraged transaction. The expected return is higher, but so too is the standard deviation.

Using equation (2), the standard deviation of her portfolio is:

= (1.4706) (34.29%) = 50.43%.

Obviously, Barb is exposed to much higher risk than Liza. The little black diamond represents Barb’s portfolio in mean/variance space.

**Buying on Margin**

In practice, a leveraged transaction such as Barb’s investment is quite straightforward. You can borrow up to a maximum of 50% of your total investment from your broker, typically at close to the risk-free rate (it is based on the broker’s call rate).

So, if you put up $100 of your own money, you can invest up to $200, by borrowing $100 from your broker: i.e. , and 1-*w* = -1.

This is called a margin transaction.

In this margin transaction, the amount of your own money relative to the total position is called the percentage margin. The current legal limit on initial marginis 50%. At least 50% of the total position must be your own (unborrowed) money.

There is also something called the maintenance margin, which is the percentage margin, or your equity in the total position to be maintained at all times. If you fall below the maintenance margin, the broker (or the exchange) will issue a margin call and ask you to bring the equity in your position back to the initial margin.

**Optimal portfolio choice**

Here, we move into Step 2 of our decision making process. Having mapped out the feasible set of portfolios, we need to choose the one that maximizes our investor’s utility. Let’s do this graphically first, and then proceed to the calculus.

Let’s first form portfolios with varying weights on both our assets, and find the utility for each such portfolio (after applying equations (1) and (2) repeatedly for each portfolio). Note that we have assumed a coefficient of risk aversion of A=3.0:

|  |  |  |
| --- | --- | --- |
| **Utility from portfolio with A =** | 3.0 |  |
| **w** | **1-w** | **E( r)** | **sigma( r)** | **Utility** |
| 0% | 100% | 5.00% | 0.00% | 5.00% |
| 25% | 75% | 9.25% | 8.57% | 8.15% |
| 48% | 52% | 13.19% | 16.53% | **9.10%** |
| 50% | 50% | 13.50% | 17.15% | 9.09% |
| 60% | 40% | 15.20% | 20.57% | 8.85% |
| 75% | 25% | 17.75% | 25.72% | 7.83% |
| 100% | 0% | 22.00% | 34.29% | 4.36% |
| 125% | -25% | 26.25% | 42.86% | -1.31% |
| 147% | -47% | 30.00% | 50.43% | -8.14% |
| 150% | -50% | 30.50% | 51.44% | -9.18% |
| 175% | -75% | 34.75% | 60.01% | -19.26% |
| 200% | -100% | 39.00% | 68.58% | -31.55% |

We can see that replacing the risk-free asset with the risky asset increases utility for a while … then, the additional expected returns from adding the risky asset are more than wiped out by the higher standard deviation associated with increasing risk. This point is better made by graphing the numbers from the above table.



The maximum utility of U=0.0910 is reached when the fraction of the risky asset is a little less than 50%.

We can solve this directly and exactly through calculus or by using Solver in Excel.

Using calculus:

Our utility function is: U = E(R) – ½ A σ2

(1A): E(Rp) = Rf + w (E(R) – Rf)

(2): σP = w ∙ σ 🡪 σp2 = w2 σ2

Substitute Equation (1A) and Equation (2) into the utility function to get

U = Rf + w (E(R) – Rf) – ½ A w2 σ2

The goal here is to maximize utility. So, we need to take the first derivative of the utility function w.r.t. the weight in the risky asset, set it equal to zero, and solve for that weight.

0 = (E(R) – Rf) – (.5) (2) A w σ2 🡪 w = 

In our example, w = .22-.05 / (3)(.3429)2 = 48.19%

So, utility is maximized when we put 48.19% of our money in the risky asset and

1 - .4819 = 51.81% in the risk-free asset.

What is the expected return and standard deviation of this *optimal portfolio*?

Once again, we need to turn to equations (1) and (2)

E(RP) = w∙ E(R) + (1 – w) ∙ Rf = (0.4819) (.22) + (0.5181) (.05) = 13.19%

and

= (.4819) (34.29%) = 16.53%

Our level of utility is:

U = .1319 – (1/2) (3) (.1653)2 = 9.097%

If we want to use Solver, we enter the utility function into Excel and use Solver to change the weights so that it maximizes the utility.

**Differential Lending and Borrowing Rates**

In the above analysis, we assumed that individuals can lend *and borrow* at the risk-free rate. Lending is fine, but what about borrowing? Can we really borrow at the risk-free rate?

Actually, the risk-free lending and borrowing rates are much closer to each other than you might imagine, and so we will generally assume that they are the same.

However, when the borrowing rate is higher than the lending rate, the reward-to-risk ratio for those borrowing is lower. This leads to a kinked CAL (kinked at the location of the risky asset). This CAL is a combination of two CALs – one with Rf Lending and one with Rf Borrowing.

In the example illustrated below, the risk-free rate for lending is 5% and the risk-free rate for borrowing is 7%. This means that the slope of the CAL (the Sharpe Ratio for the risky asset) is = 22% - 5% / 34.29% = .496 on the lending (left) side of the line and its slope is = 22% - 7% / 34.29% = .437 on the borrowing (right) side of the line.

