**Bonds with Embedded Options**

Until now, we’ve assumed that the cash flows of a bond are known with certainty (subject only to default risk). The problem though is that for bonds with embedded options, changes in interest rates affect the cash flows of the bond and we don’t know what the future interest rates will be when we buy the bond.

We need to model an environment where interest rates can change in the future and those changes will affect the cash flows. We can then consider the probabilities and amounts of the various cash flows when valuing the bond.

We need to begin with some assumptions:

1. The on-the-run (or some actively traded) issues for a given benchmark are fairly priced. We must assume that something is fairly priced and calibrate our model to it. All models of valuation are based on some value that is believed to be correct outside the model.
2. There is an assumed volatility for interest rates. We will typically use standard deviation, but the specific metric is unimportant. What is important is that we have some subjective view of how volatile interest rates will be and that we can use it to project expected future interest rates. We learned earlier that we can use historic volatility to project expected future volatility of interest rates, or we can use the implied volatility found in publicly traded options on interest rate futures contracts. Note that this is a subjective assumption and you should use sensitivity analysis to check your results.

Binomial Model – Interest rates can take one of two values next period (high or low)

Trinomial Model – Interest rates can take one of three values next period

“Multinomial Models” can take on more than three

Single factor models – only one interest rate

Multi factor models – allow for more than one interest rate (typically short-term and long-term)

We will look at a binomial single-factor model. The same principals used in this model can be applied to more complex models.

Interest Rate Tree – graphical presentation of possible interest rate paths over time

* Need discrete time periods
* Need a rate generating model – we will use a log-normal random walk with an assumed volatility
* Need a starting interest rate (today’s one-period spot rate)
* Need to accurately price the benchmark (widely traded) issue that we will be using to calibrate our model

The Interest Rate Tree will be used to discount cash flows and determine if an embedded option will be exercised.

First step in valuing a bond with embedded options is to be able to value an option-free bond.

1. Note the observed maturities, coupon rates, prices, and yields for some liquidly traded securities – the ones we assume are accurately priced.
2. Use bootstrapping to get the spot rates (zeros)
3. Use the zeros to find the implied forward interest rates. Note that the spot rates are simply the geometric averages of the implied forward rates.
4. Use the first spot rate and the implied forward rates to discount the cash flows of the option-free bond that you want to value (price). Important: This is the correct price for this bond because the cash flows are known and we have established that these are the correct discount rates because they were derived from bonds that are accurately priced.
5. We will use this bond (which we have the correct price for) to test our binomial model. Since we have the correct price on this bond, if the model comes up with a different price, the model is wrong, not the bond.

Binomial Model

**Example**: To make things simpler, we will use annual coupon payments.

Suppose we have the following existing bonds which we assume are accurately priced:

|  |  |  |
| --- | --- | --- |
| **Maturity** | **Yield** | **Mkt. Value** |
| 1 yr | 3.50% | 100 |
| 2 yrs | 4.00% | 100 |
| 3 yrs | 4.50% | 100 |

First, we use the bootstrapping technique to find the zero-coupon spot rates (note that for the three bonds above, since they sell at par, their coupon rate equals their yield). Then we find the implied forward rates.

|  |  |
| --- | --- |
| Zero-Coupon Spot Rate | **Implied Forward Rate** |
| Z1 | 3.50% |  |  |
| Z2 | 4.01% | 1f2 | 4.523% |
| Z3 | 4.53% | 2f3 | 5.580% |

If we have a 5.25% option-free bond with 3 years remaining till maturity, we can calculate the price two ways. We can discount the cash flows at the zero-coupon spot rates, or we can discount them at the implied forward rates. Of course, you will get the same price either way since the implied forward rates are derived from the zero-coupon spot rates.

Price = 5.25 + 5.25 + 105.25

 1.035 (1.041)2 (1.0453)3

 OR

Price = 5.25 + 5.25 + 105.25

 1.035 (1.035)(1.04523) (1.035)(1.04523)(1.05580)

 = 102.075

Think of the implied forward rate as being the market’s consensus of the expected future rate. Further, since we don’t know what the future rate will be, we have a distribution of possibilities, not just one expectation. We will assume that the rate will evolve over time based on a lognormal random walk, with the volatility measured by σ. Further, we will assume, that based on past observations, we calculate the value of this volatility as .10 (σ = .10).

We will consider that each year, interest rates will either be high or low and that the current rates will help us determine what the future rates will be. This allows us to generate the Binomial Interest-Rate Tree. If rates are low, they will be rL. If rates are high, they will be rH, where rH = rL (e2σ). So we need to come up with an expectation for rL, and that will generate for us a value for rH.

Rates for the first year are r0 and we know that they are 3.5%.

Note that r0 = z1

Rates for the second year are r1L and r1H.

Note that r1 = 1f2

Rates for the third year are r2LL, r2HL and r2HH.

Note that r2LH = r2HL.

Note that r2 = 2f3.

The value of the bond at time 0 will be the discounted value of the bond at time 1 plus the discounted value of the coupon payment at time 1.

We know that the coupon payment will be 5.25, and we know that we should discount it at 3.5%, but we don’t know what the value of the bond will be at time 1.

It’s value at time 1 will be the (time 1) discounted value of the market value at time 2 plus the discounted value of the time 2 coupon payment.

The problem here is that we don’t know what interest rates will be, so we don’t know what rate to discount the time 2 value at.

Further, we don’t know what the time 2 value will be, because we don’t know the time 3 interest rates. Fortunately, though, we do know the time 3 values because the bond matures then.

To find r1L and r1H, we have to look at the 2-year, 4% observable bond.

We know that its cash flow in time 2 will be 104.

We will use trial and error (Solver) to find the value of r1L and use that value to find r1H.

We are looking for values that will cause the current price to be 100 (which we observe).

If Solver tells us to use 4.074% for r1L, this will correspond to 4.976% for r1H

because 4.976% = 4.074% (e.1x2).

Discounting 104 at each rate gives us:

 104 = 99.929

1.04074

 104 = 99.071

1.04976

So our time 1 value for the bond will be either 99.929 or 99.071.

Our time 0 value will be the discounted value of the time 1 value plus the coupon payment – discounted at the known rate of 3.5%.

99.929 + 4 = 100.415

 1.035

99.071 + 4 = 99.585

 1.035

Our time 0 value will either be 100.415 or 99.585 with equal probability. If we average these two values together, we get the price of 100 that we were looking for.

So, investors know that z1 is 3.5%, but don’t know what 1f2  will be. The expectation is that if interest rates are low, it will be 4.074% and if interest rates are high, it will be 4.976%. And the expectation is that these are equally likely.

This must be so, because these rates equate the price of the correctly priced, observable bond with the present value of its cash flows.

Now that we know the values of r1L and r1H, we can find the values of r2LL, r2LH, and r2HH.

To do this, we need to use the (correctly priced) observed 3-year 4.5% bond and find (by using solver), values that will solve the following:







V2HH = 104.5

 1+r2LL

V2HL = 104.5

 1+r2LH\*e.2σ

V2LL = 104.5

 1+r2HH\*e.4σ

With σ = 10%, we use solver to find that r2LL = 4.53% and we find that:

r2LH = 5.532%

r2HH= 6.757% and then:

V2LL = 97.886

V2LH = 99.022

V2HH = 99.972

V1L = 98.074

V1H = 99.926

And the price of the bond is 100 which is exactly what it was before we introduced interest rate volatility into the equation.

Now that we have all the interest rates, we can use them to come up with a Binomial Interest Rate Tree for our 3-year 5.25% option-free (non-callable) bond – paying particular attention to the values at each node.

V2LL = 105.25 = 98.588

 1.06757

V2LH = 105.25 = 99.732

 1.05532

V2HH = 105.25 = 100.689

 1.0453





V1L = 99.461

V1H = 101.333



Price = 102.075 for a non-callable bond

Note that in Excel, cells B32 and B33 confirm this price by discounting at the zeros and at the implied forward rates. So our model is confirmed.

Now that we have the price of a non-callable bond, we can price a callable bond.

Start with the far right and replace any values that are greater than the call price with the call price and then work towards the left. This is because we assume that the bond will be called in those interest-rate environments.

A word of caution here: There may be some subjectivity in determining whether or not the issuer will call the bond. The analyst needs to determine when it is a positive NPV decision for the issuer to call the bond – considering taxes, floatation costs, etc.

After replacing the first value, rework all the others. If any to the left of the one we replaced is still greater than the call price, replace it as well.

Once we have all values less than or equal to the call price, we can come up with the price of the callable bond.

We first replace the 100.689 value for V2HH with 100 (the call price). This changes V1H to 101.002 and the price of the bond to 101.915.

But now we need to replace the 101.002 in V1H with 100 also. This now gives us a price for the callable bond of 101.431.

Can the bond be called at time 0? If so, change the prices there as well. In this example, we will say that the bond is call-protected until time 1.

If the spot rates and implied forward rates were those for the same issuer (corporation), the price we come up with (101.431) is the appropriate price of a callable bond for that issuer.

If the theoretical spot rates and implied forward rates are for Treasuries, the price we come up with (101.431) is the appropriate price of a callable Treasury and we still need to consider credit risk.

In any event, the value of the call option is:

 102.075

-101.431

 0.644

This value of the call option needs to be subtracted from the price of the option-free corporate bond to come up with the price of the callable corporate bond. Note that the value of the call option is the same, whether the bond is a Treasury or a Corporate bond.

If we increase the volatility assumption from 10% to 20%, notice that the price of the option-free bond doesn’t change. This makes sense because interest-rate volatility does not affect its cash flows.

But the price of the callable bond drops to 100.986. This is because the call option has more value when volatility increases (remember from options classes). If the option (which the issuer holds) has more value, the bond has less value.

101.431 (at 10% assumed volatility) and 100.986 (at 20% assumed volatility) are model-driven prices for our callable bond. The actual market price will depend on what investors are willing to buy and sell it for (depending on what they assume the future volatility will be, when the bond might be called, etc.). The price also depends on what level of volatility investors expect and what interest rate model they use (remember that this is just one simple model – there are others).

Assume that you observe that this bond has a market price of 101. Is this bond a good buy?

The answer is: It depends.

Comparing it to the price generated by an option-free model, it is a great buy at 101 because the model says 102.075 is the correct price.

Comparing it to the price generated by the model for a callable bond with a 10% volatility assumption, it is a good buy because that model says that 101.431 is the correct price.

But comparing it to the price generated by the model for a callable bond with a 20% volatility assumption, it is slightly overpriced because the model says that 100.986 is the correct price.

So the question is…. What do you think the volatility of interest rates will be, and are we modeling interest rate movements correctly?

Option Adjusted Spread

We don’t usually talk about the bond being overpriced or underpriced in terms of its price, but in terms of its spread.

The OAS is the constant spread that, when added to z1 and all the rates on the binomial interest rate tree will cause the market price of the bond being valued to be equal to the present value of its cash flows. This is telling us if the bond gives us more yield than the model or less yield than the model. We need to use solver to find the OAS.

With a 10% volatility assumption, we need to add 0.23% to each interest rate (r0, r1H, r1L, r2HH, r2HL, and r2LL), to get the present value of the cash flows in our callable bond model to equal the market price of the bond of 101. So 0.23% is the OAS for this bond (with 10% assumed volatility).

With a 20% volatility assumption, we actually need to **subtract** .01% from each interest rate to get the present value of the cash flows to equal the market price of 101.

Notice that when the volatility assumption increased, the OAS decreased from .23% to

-.01%. So depending on your assumption, the bond is either offering you a .23% spread over what it should (what our model says), or giving you a .01% less spread than it should.

If the OAS = 0.00%, the market price of the bond is what our model predicts.

Effective Duration and Effective Convexity

We can use the same binomial interest rate tree to model how duration and convexity are affected by a change in the bond’s cash flows due to a change in interest rates.

When we calculated duration and convexity before, we assumed that the cash flows were independent of any changes in interest rates. So we calculated a change in the price of a bond due to a change in interest rates with the cash flows of the bond remaining constant.

As we know, if interest rates change, the cash flows of a bond with embedded options can change. Effective duration and convexity include this in their model.

Effective duration is found by changing interest rates in the binomial interest rate tree by a fixed amount (both up and down) and observing how much the modeled price of the bond changes. We take the average of that change and express it as a percentage of the bond’s value. Effective convexity is found in a similar way.

Interestingly, a callable bond can exhibit “negative convexity”. This is where the price of the bond goes down when interest rates go down instead of going up as we have always learned. The intuitive explanation is that if interest rates go down, it may make it more likely that the bond will be called, thus reducing its expected cash flows and causing the price of the bond to go down.